Series-Parallel Duality in the Mathematics of Appraisal

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It has long been noticed that the basic functions in the mathematics of real estate appraisal and in financial arithmetic come in pairs where one function is the reciprocal of the other. For instance, the "payments to amortize a principal of one" is the reciprocal of the "principal amortized by payments of one." This paper shows that this phenomenon is an example of the series-parallel duality ordinarily studied in electrical circuit theory and combinatorial mathematics.

Introduction

In financial arithmetic and in the real estate appraisal literature, it has been noticed that the basic formulas occur in pairs, one being the reciprocal of the other. For instance, one popular text on real estate appraisal presents the "Basic Functions of Compound Interest and Their Reciprocals" [Friedman and Ordway 1988, 70]. The functions could be presented as follows to bring out the underlying symmetry.

Function
Principal Retired by Payment of One $(1+r)^{-n}$
Principal Amortized by Payments of One $a(n,r)=\frac{1}{(1+r)^1}+\frac{1}{(1+r)^2}+\Lambda+\frac{1}{(1+r)^n}$
Fund Accumulated by One per Period $s(n,r)=(1+r)^{n-1}+\ldots+(1+r)^1+1$
Reciprocal
Payment to Retire Principal of One $(1+r)^{-n}$
Payments to Amortize a Principal of One $\frac{1}{a(n,r)}=(1+r)^1:(1+r)^2:\Lambda:(1+r)^n.$
Payments to Accumulate a Fund of One $\frac{1}{s(n,r)}=\frac{1}{(1+r)^{n-1}}:\Lambda\frac{1}{(1+r)^1}:1.$

The Six Functions of One

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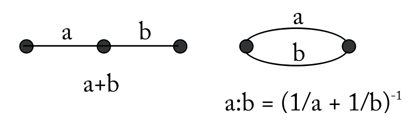
> *From the viewpoint of pure mathematics, the parallel sum is "just as good" as the series sum.*

The purpose of this paper is to show that these reciprocal formulas are an example of the series-parallel duality normally associated with electrical circuit theory. The basic mathematics is quite simple and starts with noticing that there is a certain "parallel addition" dual to the usual notion of (series) addition.

Parallel Addition

When resistors with resistances a and b are placed in series, their compound resistance is the usual sum (hereafter the *series sum*) of the resistances a+b. If the resistances are placed in parallel, their compound resistance is the *parallel sum* of the resistances, which is denoted by the full colon:

$$a:b=\left(a^{-1}+b^{-1}\right)^{-1}=\frac{1}{\frac{1}{a}+\frac{1}{b}}.$$



Series and Parallel Sums

The parallel sum is associative x:(y:z)) = (x:y):z, commutative x:y = y:x, and distributive x(y:z) = xy:xz. On the positive reals, there is no identity element for either sum but the "closed circuit" 0 and the "open circuit" ∞ can be added to form the extended positive reals. Those elements are the identity elements for the two sums, x+0 = x = x: ∞ . That is, adding a short circuit in series to a resistor does not change the resistance, and adding an open circuit in parallel to a resistor does not change the resistance.

For fractions, the series sum is the usual addition expressed by the annoyingly asymmetrical rule: "Find the common

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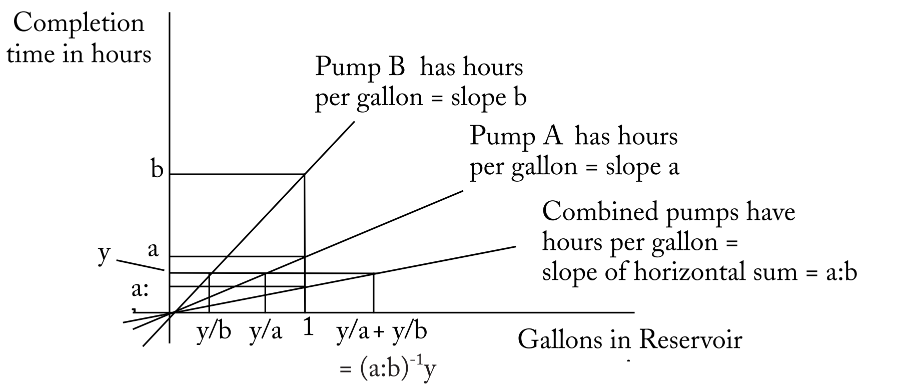


Horizontal Sum of Lines

denominator and then add the numerators." The parallel sum of fractions restores symmetry since it is defined in the dual fashion: "Find the common numerator and then (series) add the denominators."

$$\frac{a}{b} : \frac{a}{d} = \frac{a}{b+d}$$

The usual series sum of fractions can also be obtained by finding the common *numerator* and then taking the *parallel* sum of the denominators.

$$\frac{a}{b} + \frac{a}{d} = \frac{a}{b:d}$$

The parallel sum of fractions can also be obtained by finding the common denominator and taking the *parallel* sum of numerators.

$$\frac{a}{b} : \frac{c}{b} = \frac{a:c}{b}$$

The rules for series and parallel sums of fractions can be summarized in the following four equations which restore full symmetry.

$$\frac{a}{1} + \frac{b}{1} = \frac{a+b}{1} \qquad \frac{a}{1} : \frac{b}{1} = \frac{a:b}{1}$$

$$\frac{1}{a} : \frac{1}{b} = \frac{1}{a+b} \qquad \frac{1}{a} + \frac{1}{b} = \frac{1}{a:b}$$

Series Chauvinism

From the viewpoint of pure mathematics, the parallel sum is "just as good" as the series sum. It is only for empirical and perhaps even some accidental reasons that so much mathematics is developed using the series sum instead of the equally good parallel sum. There is a whole "parallel mathematics" which can be developed with the parallel sum replacing the series sum. Since the parallel sum can be defined in terms of the series sum (or vice-versa), "parallel mathematics" is essentially a new way of looking at certain known parts of mathematics.

Exclusive promotion of the series sum is "series chauvinism." Before venturing further into the parallel universe, we might suggest some exercises to help the reader combat the heritage of series chauvinism. Anytime the series sum seems to occur naturally in mathematics with the parallel sum nowhere in sight, it is an illusion. The parallel sum lurks in a "parallel" role that has been unfairly neglected.

> *The fundamental intuition is that the series-parallel duality gives reciprocals*

For instance, a series chauvinist might point out that the series sum appears naturally in the rule for working with exponents $x^a x^b = x^{a+b}$ while the parallel sum does not. But this is only an illusion due to our mathematically arbitrary symmetry-breaking choice to take exponents to represent powers rather than roots. Let a pre-superscript stand for a root (just as a post-superscript stands for a power) so ${}^2x$ would be the square root of x. Then the rule for working with these exponents is ${}^a{}^b x = {}^{a:b}x$ so the parallel sum does have a role symmetrical to the series sum in the rules for working with exponents.

Parallel Sums in School Math

In high school algebra, parallel sums occur in the computation of completion times when activities are run in parallel. If pump A can fill a reservoir in a hours and pump B can fill the same reservoir in b hours, then running the two pumps simultaneously will fill the reservoir in a:b hours. The two pumps can be represented as straight lines through the origin and the combined pump is represented by the parallel or horizontal sum of the lines. The slope of the horizontal sum of two positively sloped straight lines is the parallel sum of the slopes.

For any given y, y/b is the x coordinate so the y = bx, i.e., x = y/b = gallons pumped by pump B in y hours, and y/a is the x coordinate so that y = ax, i.e., x = y/a = gallons pumped by A in y hours. The "horizontal" sum of those two x coordinates is: $y/a + y/b = (1/a + 1/b)y = (a:b)^{-1}y$ which is the x coordinate that gives the same y on the line y = (a:b)x, i.e., x = y/(a:b) = gallons pumped by both pumps in y hours. Thus y = (a:b)x is the number of hours it takes (completion time) to pump x gallons when the pumps are used in parallel.

The Harmonic Mean

The *harmonic mean* of n positive reals is n times their parallel sum. Suppose an investor spends $100 a month for two months buying shares in a certain security. The shares cost $5 each the first month and $10 each the second month. What is the average price per share purchased? At first glance, one might average the $5 and $10 prices to obtain an average price of $7.50—but that would neglect the fact that twice as many shares were purchased at the lower price. Hence one must compute the weighted average taking into account the number of shares purchased at each price, and that yields the harmonic mean of the two prices.

$$\frac{(20\times\$5)+(10\times\$10)}{30} = 2(\$5:\$10) = \$6\frac{2}{3}.$$

This investment rule is called "dollar cost averaging." A financial advisory letter explained a benefit of the method.

> *First, dollar cost averaging helps guarantee that the average cost per share of the shares you buy will always be lower than their average price. That's because when you always spend the same dollar amount each time you buy, naturally you'll buy more shares when the fund's price is lower and fewer shares when its price is higher. [Scudder Funds 1988]*

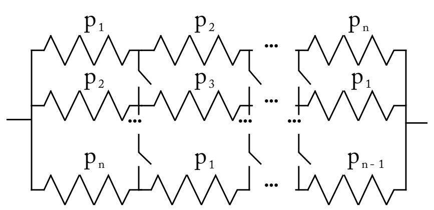
Let $p_1, p_2, \ldots, p_n$ be the price per share in each of n time periods. The average cost of the shares is the harmonic mean of the share prices, $n(p_1 : p_2 : \ldots : p_n)$, and the average price is just the usual arithmetical mean of the prices, $(p_1 + p_2 + \ldots + p_n)/n$.

The inequality that the average cost of the shares is less than or equal to the average of the share prices follows from

$(A:B) + (C:D) \le (A+C) : (B+D)$
Lehman's Series-Parallel Inequality

for positive A, B, C, and D [1962; see Duffin 1975].

This application of the series-parallel inequality can be seen by considering the prices as resistances in the following diagram (note how each of the n rows and each of the n columns contains all n resistances or prices).



Intuitive Proof of Lehman's Inequality

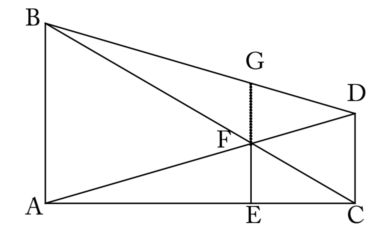
When all the switches are open, the compound resistance is the parallel sum (n times) of the series sum of the prices, which is just the arithmetical mean of the prices. When the switches are closed, the compound resistance is the series sum (n times) of the parallel sum of the prices, which is the harmonic mean of the prices. Since the resistance is smaller or the same with the switches closed, we have for any positive $p_1, p_2, \ldots, p_n$,

> *The parallel sum "smooths" balloon payments to yield the constant amortization payment to pay off a loan.*

$$\mathrm{HarmonicMean} = n[p_1 : p_2 : \ldots : p_n] \le \frac{\sum_{i=1}^{n} p_i}{n} = \mathrm{ArithmeticMean}$$

Geometric Representation of the Parallel Sum

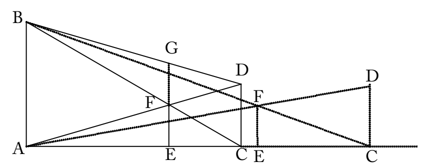
The harmonic mean of two numbers is twice their parallel sum, just as the usual series or arithmetical mean is half their series sum. For the geometric representation of the harmonic mean and parallel sum, draw a line FG through the point E where the diagonals cross in the trapezoid ABDC.



EF = AB:CD

Then EG is the harmonic mean of parallel sides AB and CD, i.e., EG = 2(AB:CD). Since F bisects EG, the distance EF is the parallel sum of AB and CD, i.e., EF = AB:CD.

It is particularly interesting to note that the distance AC is arbitrary. If CD is shifted out parallel to C'D' and the new diagonals AD' and BC' are drawn (as if rubber bands connected B with C and A with D), then the distance E'F' is again the parallel sum of AB and CD (= C'D').



EF = E'F' = AB:CD where CD = C'D'

Series-Parallel Duality

The Reciprocity Map

The duality between the series and parallel additions on the positive reals R+ can be studied by considering the (bijective) reciprocity map

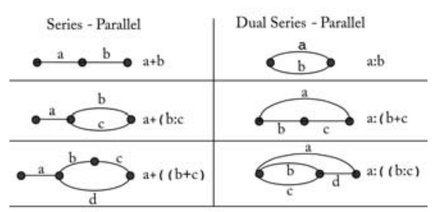
$\rho: R^+ \to R^+$ defined by $\rho(x) = 1/x$.

The reciprocity map preserves the unit $\rho(1) = 1$, preserves multiplication $\rho(xy) = \rho(x)\rho(y)$, and *interchanges* the two additions:

$\rho(x+y) = \rho(x):\rho(y)$ and $\rho(x:y) = \rho(x)+\rho(y)$.

The reciprocity map captures series-parallel duality on the positive reals.

Much of the previous work on series-parallel duality has used methods drawn from graph theory and combinatorics [e.g., MacMahon 1881, Riordan and Shannon 1942, Duffin 1965, and Brylawski 1971]. MacMahon called a series connection a "chain" and a parallel connection a "yoke" (as in ox yoke). A series-parallel network is constructed solely from chains and yokes (series and parallel connections). By interchanging the series and parallel connections, each series-parallel network yields a dual or conjugate series-parallel network. To obtain the dual of an expression such as a+b, apply the reciprocity map to obtain (1/a):(1/b) but then, for the atomic variables, replace 1/a by a and so forth in the final expression. Hence the MacMahon dual to a+b would be a:b, and the dual expression to a+((b+c):d) would be a:((b:c)+d) (see below).



Conjugate Series-Parallel Networks

If each variable a, b, ... equals one, then the reciprocity map carries each expression for the compound resistance into the conjugate expression. Hence if all the "atomic" resistances are one ohm, a = b = c = d = 1, and the compound resistance of a series-parallel network is R, then the compound resistance of the conjugate

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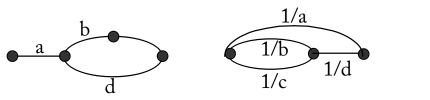
network is 1/R [MacMahon 1881, 1892]. With any positive reals as resistances, MacMahon's chain-yoke reciprocity theorem continues to hold if each atomic resistance is also inverted in the conjugate network (i.e., if we just apply the reciprocity map and do not make the extra inversion of the atomic variables).



a+((b+c):d (1/a):((1/b):(1/c))+(1/d) = 1/R

MacMahon Chain-Yoke Reciprocity Theorem

The theorem amounts to the observation that the reciprocity map interchanges the two sums while preserving multiplication and unity. The fundamental intuition is that the series-parallel duality gives reciprocals. We now apply series-parallel duality to the type of mathematics used in valuation to show that it is the source of the reciprocal functions in the appraisal literature.

Dual Equations on the Positive Reals

Any equation on the positive reals concerning the two sums and multiplication can be dualized by applying the reciprocity map to obtain another equation. The series sum and parallel sum are interchanged. For example, the equation

$$\frac{1}{3}\left(5+\frac{2}{5}+\frac{3}{5}\right)=2$$

dualizes to the equation

$$3\left(\frac{1}{5}:\frac{5}{2}:\frac{5}{3}\right)=\frac{1}{2}.$$

The following equation

$$1=(1+x):\left(1+\frac{1}{x}\right)$$

holds for any positive real x. Add any x to one and add its reciprocal to one. The results are two numbers larger than one and their parallel sum is exactly one. Dualizing yields the equation

$$1=\left(1:\frac{1}{x}\right)+(1:x)$$

for all positive reals x. Taking the parallel sum of any x and its reciprocal with one yields two numbers smaller than one which sum to one.

For any set of positive reals $x_1,\ldots,x_n$, the parallel summation can be expressed using the capital P:

$$\overset{n}{\underset{i=1}{\mathrm{P}}}x_i=\left(\sum_{i=1}^{n}x_i^{-1}\right)^{-1}.$$

Parallel Summation

Series and Parallel Geometric Series

The following formula (and its dual) for partial sums of geometric series (starting at i = 1) are useful in financial mathematics (where x is any positive real).

$$\sum_{i=1}^{n}(1:x)^i=(1:x)\sum_{i=0}^{n-1}(1:x)^i=$$
$$(1:x)\frac{\left(1-(1:x)^n\right)}{1-(1:x)}=x\left(1-(1:x)^n\right)$$

Partial Sums of Geometric Series

Dualizing yields a formula for partial sums of parallel-sum geometric series. The dual of the series subtraction a–b where a > b is the parallel subtraction x\y = $[1/x-1/y]^{-1}$ where x < y.

$$\overset{n}{\underset{i=1}{\mathrm{P}}}(1+x)^i=(1+x)\overset{n-1}{\underset{i=0}{\mathrm{P}}}(1+x)^i=$$
$$(1+x)\frac{\left(1\backslash(1+x)^n\right)}{1\backslash(1+x)}=\frac{x}{1-(1+x)^{-n}}.$$

Partial Sums of Dual Geometric Series

Dualization can also be applied to infinite series. Taking the limit as $n\to\infty$ in the above partial sum formulas yields for any positive reals x the dual summation formulas for series and parallel sum geometric series that begin at the index i = 1.

$$\sum_{i=1}^{\infty}(1:x)^i=x=\overset{\infty}{\underset{i=1}{\mathrm{P}}}(1+x)^i$$

The parallel sum series in the above equation can be used to represent a repeating decimal as a fraction. An example will illustrate the procedure so let z = .367367367… where the "367" repeats. Then since 1/a + 1/b = 1/(a:b), we have:

$$z=.367367K=\sum_{i=1}^{\infty}\frac{367}{(1000)^i}=\frac{367}{\overset{\infty}{\underset{i=1}{\mathrm{P}}}(1000)^i}.$$

Taking y = x+1 for x > 0 in the previous geometric series equation yields

$$\overset{\infty}{\underset{i=1}{\mathrm{P}}}y^i=y-1$$

for y > 1 which is applied to yield

$$z=.367367K=\frac{367}{\overset{\infty}{\underset{i=1}{\mathrm{P}}}(1000)^i}=\frac{367}{1000-1}=\frac{367}{999}$$

For any positive real x, the beautiful dual formulas for the geometric series with indices beginning at i = 0 can be obtained by serial or parallel adding 1= (1:x)0 = (1+x)0 to each side.

$$\sum_{i=0}^{\infty}(1:x)^i=(1+x)$$

Geometric Series for any Positive Real x

$$\overset{\infty}{\underset{i=0}{\mathrm{P}}}(1+x)^i=(1:x).$$

Dual Geometric Series for any Positive Real x

Series-Parallel Duality in Financial Arithmetic

Parallel Sums in Financial Arithmetic

The parallel sum has a natural interpretation in finance so that each equation and formula in financial arithmetic can be paired with a dual equation or formula. The parallel sum "smooths" balloon payments to yield the constant amortization payment to pay off a loan. If r is the interest rate per period, then $PV(1+r)^n$ is the one-shot balloon payment at time n that would pay off a loan with the principal value of PV. The similar balloon payments that could be paid at times t =1, 2,..., n, any one of which would pay off the loan, are

$PV(1+r)^1$, $PV(1+r)^2$, ..., $PV(1+r)^n$.

But what is the equal amortization payment PMT that would pay off the same loan when paid at each of the times t =1, 2, ..., n? It is simply the parallel sum of the one-shot balloon payments:

$$PMT=PV(1+r)^1:PV(1+r)^2:\ldots:PV(1+r)^n$$
$$=\overset{n}{\underset{i=1}{\mathrm{P}}}PV(1+r)^i.$$

Amortization Payment is Parallel Sum of Balloon Payments

How does the total amount of money paid with equal loan payments compare with the one-time balloon payments? The sum of all the amortization payments nPMT is the harmonic mean of the balloon payments.

This use of the parallel sum is not restricted to financial arithmetic. For example, suppose a forest of initial size PV (in harvestable board feet) grows at the rate r_i in the i^{th} period. Then

$$P_m=PV\prod_{i=1}^{m}(1+r_i)$$

would be the *one-shot harvest* that could be obtained at the end of the m^{th} period. For instance, P_3, P_{17}, and P_{23} are the amounts that could be harvested if the whole forest was harvested at the end of the 3rd, 17th, or the 23rd period. But what is the smooth or equal harvest PMT so that if PMT was harvested at the end of the 3rd, 17th, and the 23rd periods, then the forest would just be completely harvested at end of that last period? That smooth harvest amount is just the parallel sum of the one-time harvests:

$PMT = P_3 : P_{17} : P_{23}$.

The total harvest under the equal harvest method is the harmonic mean of the three one-shot harvests.

Returning to financial arithmetic, the discounted present value at time zero of n one dollar payments at the end of periods 1, 2,…, n is a(n,r), *the present value of an annuity of one.*

$$a(n,r)=\frac{1}{(1+r)^1}+\frac{1}{(1+r)^2}+\Lambda+\frac{1}{(1+r)^n}.$$

Present Value of Payments of One

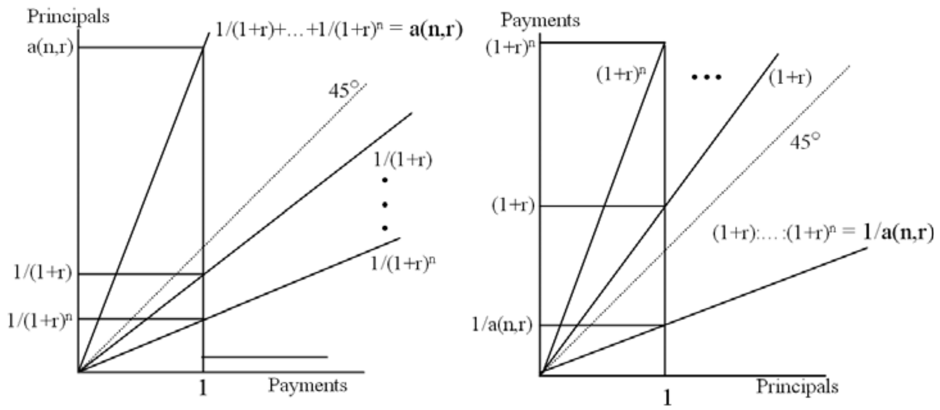
Dualizing [i.e., applying the reciprocity map and using the fact that ρ(1/1+r) = 1+r] yields:

$$\frac{1}{a(n,r)}=(1+r)^1:(1+r)^2:\Lambda:(1+r)^n.$$

Payments to Amortize a Principal of One

For the principal value of one dollar at time zero, the one-shot payments at times 1,2,…, n that would each pay off the principal are the compounded principals $(1+r)^1$, $(1+r)^2$,…, $(1+r)^n$. The parallel sum $(1+r)^1$: $(1+r)^2$ paid at times 1 and 2 would pay off the $1 principal. Similarly, the parallel sum of the first three one-shot payments paid at times 1, 2, and 3 would pay off the $1 principal, and so forth.

Geometrically, a positive real number could be represented by a straight line through the origin with that positive slope. The series sum of the reals would geometrically be the vertical sum of the lines while the parallel sum would be the horizontal sum of the lines. Dualization would simply be flipping the graph around the 45° line.



Illustrations of Primal and Dual Identities

Suppose the constant interest rate is 20 percent per period. Then the discounted present value of two amortization payments of 1 at the end of the first and second period is principal value of the loan paid off by those payments, i.e., 55/36:

$$a(2,.20)=\frac{55}{36}=\frac{1}{(1.2)^1}+\frac{1}{(1.2)^2}.$$

The equation dualizes to:

$$\frac{1}{a(2,.20)}=\frac{36}{55}=(1.2)^1:(1.2)^2.$$

The amounts $(1.2)^1$ = 6/5 and $(1.2)^2$ = 36/25 are the compounded principal values of a $1 loan so they are the one-shot or balloon payments that would pay off a loan of principal value $1 if paid, respectively, at the end of the first or the second period. Their parallel sum, 36/55, is the equal amortization payment that would pay off that loan of $1 if paid at the end of both the first and second periods.

These facts can be arranged in the following dual format.

Primal Fact:

The series sum of the discounted amortization payments for a loan is the principal of the loan.

Dual Fact:

The parallel sum of the compounded principals of a loan is the amortization payment for the loan.

The example illustrates some of the substitutions involved in dualizing the interpretation.

series sum	⇔	parallel sum
discounting	⇔	compounding
principals	⇔	payments

Future Values and Sinking Fund Deposits

Another staple of financial arithmetic is the computation of sinking fund deposits. The compounded future value at time n of n one dollar deposits at times 1,2,…, n is s(n,r), *the accumulation of one per period.*

$s(n,r)=(1+r)^{n-1}+(1+r)^{n-2}+\Lambda+(1+r)^1+1=a(n,r)(1+r)^n$.

Fund Accumulated by One per Period

The discounted values $1/(1+r)^{n-1}$,…, 1/(1+r), 1 of a one-dollar fund are the one-shot deposits at times 1,…,n 1, n that would each by itself yield a one-dollar future value for the sinking fund at time n. The parallel sum of these one-shot deposits is the (equal) sinking fund deposit at times 1,…,n-1, n that would yield a one-dollar fund at time n:

$$\frac{1}{s(n,r)}=\frac{1}{(1+r)^{n-1}}:\Lambda\ \frac{1}{(1+r)^1}:1.$$

Sinking Fund Factor: Payments to Accumulate a Fund of One

The sum of the smooth sinking fund deposits is the harmonic mean of the one-shot deposits.

The dual interpretations might be stated as follows.

The series sum of the n compounded one-dollar deposits is the sinking fund that is accumulated by the one-dollar deposits.
The parallel sum of the n discounted one-dollar funds is the deposit that accumulates to a one-dollar sinking fund.

We now have reproduced the six basic functions of the valuation literature as three pairs of series-parallel duals.

Infinite Streams of Payments

The formulas for amortization payments can be extended to an infinite time horizon. This involves a financial interpretation for the dual geometric series with indices beginning at i = 1:

$$\sum_{i=1}^{\infty}(1:x)^{i} = x = \mathop{\mathrm{P}}_{i=1}^{\infty}(1+x)^{i}.$$

Taking x = 1/r so that 1:x = 1:1/r = 1/(1+r) in the series summation yields the fact that the discounted present value of the constant stream of one-dollar payments at times 1, 2,… is reciprocal of the interest rate x = 1/r.

Function
Principal Retired by Payment of One $(1+r)^{-n}$
Principal Amortized by Payments of One $a(n,r)=\frac{1}{(1+r)^{1}}+\frac{1}{(1+r)^{2}}+\Lambda+\frac{1}{(1+r)^{n}}.$
Fund Accumulated by One per Period $s(n,r)=(1+r)^{n-1}+\ldots+(1+r)^{1}+1$
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Payments to Amortize a Principal of One $\frac{1}{a(n,r)}=(1+r)^{1}:(1+r)^{2}:\Lambda:(1+r)^{n}.$
Payments to Accumulate a Fund of One $\frac{1}{s(n,r)}=\frac{1}{(1+r)^{n-1}}:\Lambda\frac{1}{(1+r)^{1}}:1.$

The Six Functions of One

$$\sum_{i=1}^{\infty}(1+r)^{-i} = \frac{1}{(1+r)^{1}}+\frac{1}{(1+r)^{2}}+\Lambda = \frac{1}{r}.$$

Perpetuity Capitalization Formula

Taking x = r in the parallel summation

The series sum of the stream of discounted \$1 amortization payments (which is the principal amortized by a \$1 amortization payment) is the reciprocal of the interest rate, $(1+r)^{-1}+(1+r)^{-2}+\Lambda = r^{-1}.$
The parallel sum of the stream of compounded \$1 principals (which is the payment that amortizes a \$1 principal) is the interest rate, $(1+r)^{1}:(1+r)^{2}:\Lambda = r.$

yields the fact that the parallel sum of compounded values of one dollar is the interest rate r, the constant payment at t = 1, 2,… that pays off a principal value of one dollar. Thus the dual to the annuity capitalization formula (1/r is the principal whose payments are 1) is the fact that the constant income stream of r is the equivalent of the capital of \$1 (r is the payments whose principal is 1).
Some of the standard formulas and their duals are recapitulated in the following table.

Primal	
X	
Series Sum a+b	
$\sum_{i=1}^{n}(1:x)^{i} = x\left(1-(1:x)^{n}\right)$	
$\sum_{i=1}^{n}\left(\frac{1}{1+r}\right)^{i} = \frac{1-(1+r)^{-n}}{r} = a(n,r)$ take x = 1/r above.	
$\sum_{n=1}^{\infty}(1:x)^{n} = x$	for any positive x
$\sum_{n=1}^{\infty}\left(\frac{1}{1+r}\right)^{n} = \frac{1}{r}$	take x = 1/r above.
Dual (by Reciprocity Transformation)	
X-1	
Parallel Sum 1/a : 1/b	
$\mathop{\mathrm{P}}_{i=1}^{n}(1+x)^{i} = \frac{x}{1-(1+x)^{-n}}$	
$\mathop{\mathrm{P}}_{i=1}^{n}(1+r)^{i} = \frac{r}{1-(1+r)^{-n}} - \frac{1}{a(n,r)}$ take x = r above.	
$\mathop{\mathrm{P}}_{n=1}^{\infty}(1+x)^{n} = x$	for any positive x
$\mathop{\mathrm{P}}_{n=1}^{\infty}(1+r)^{n} = r$	take x = r above.

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